Chapter 11- Scattering Theory
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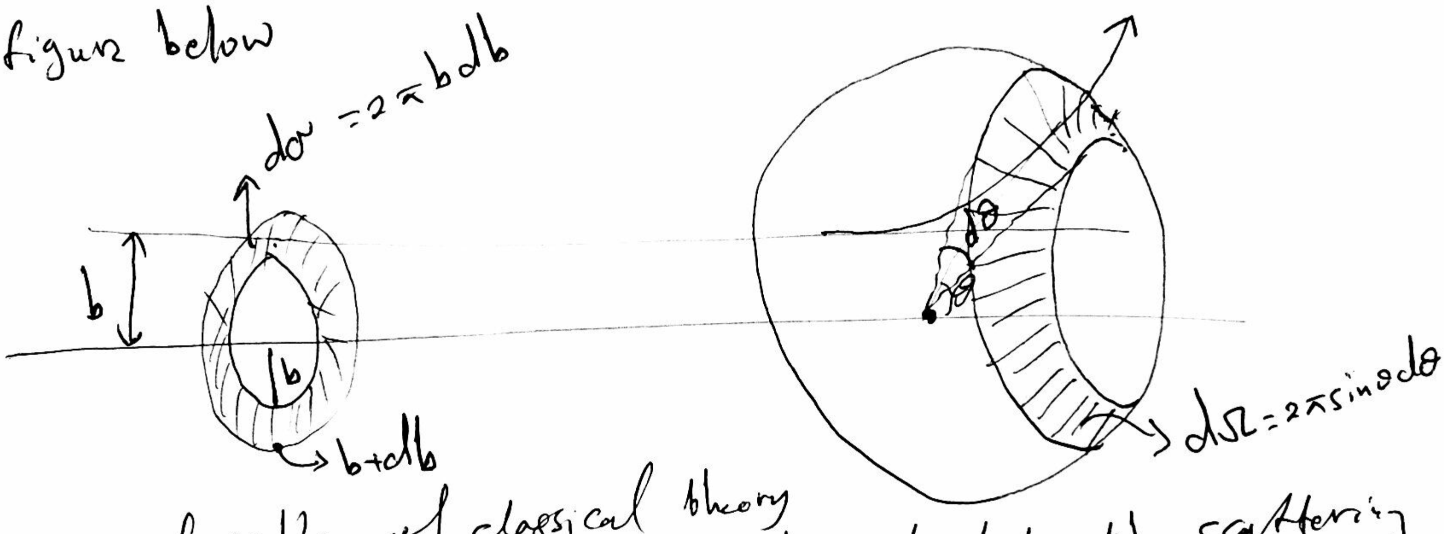
Scattering and cross section!

- classical scattering theory!

consider an incident particle on some scattering center (Target) with an energy $\underline{\underline{E}}$ and impact parameter $\underline{\underline{b}}$ and it emerges at some scattering angle θ as shown in the figure below



The essential problem of classical theory is this: given the impact parameter b, calculate the scattering angle θ i.e find $\theta(b)$. experimentally, θ is measured and $b(\theta)$ is calculated. in general, the smaller the impact parameter, the greater the scattering angle θ.

- as seen in the figure above, particles incident within an infinitesimal patch of cross section $d\sigma$ will scatter into a corresponding solid angle $d\Omega$. the ratio $d\sigma/d\Omega$ is defined as the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b\,db}{2\pi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\frac{db}{d\theta}$$

Notice that, in most cases $db/d\theta$ is negative, since θ is typically a decreasing function of b (the slope is negative), so it is common to write $d\sigma/d\Omega$ as

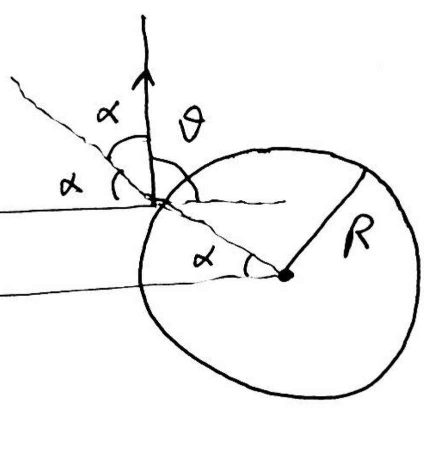
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$ as $d\sigma/d\Omega$ is always > 0

- the diff cross section is also defined as the ratio of the # of particles scattered into the direction (θ, ϕ) per unit time per unit solid angle divided by the incident flux

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{N}{J_i}$$

The total cross section is obtained by integrating over all solid angles $\Rightarrow \sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$

roughly speaking, it is the total area of incident beam that is scattered by the target $\Rightarrow$ the unit of $\frac{d\sigma}{d\Omega}$ or σ is unit of area (Note that $d\Omega$ is dimensionless quantity)



Example: Hard sphere scattering

suppose the target is a billiard ball, of radius R and the incident particle is another BB which bounces off elastically $\Rightarrow$

$$b = R\sin\alpha \quad ; \quad \theta = \pi - 2\alpha \Rightarrow \alpha = \frac{\pi - \theta}{2} = \pi/2 - \frac{\theta}{2}$$

$$\Rightarrow b = R\sin\left(\pi/2 - \frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \theta = \begin{cases} 2\cos^{-1}(b/R) & ; \ b \leq R \\ 0 & ; \ b \geq R \end{cases}$$

Now $\frac{db}{d\theta} = -\frac{1}{2} R \sin\left(\frac{\theta}{2}\right) \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$

$$= \frac{R\cos\frac{\theta}{2}}{\sin\theta}\frac{1}{2}R\sin\frac{\theta}{2}$$

$$= \frac{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\sin\theta}\frac{R^2}{2}$$

$$= \frac{R^2}{4}, \text{ using } \sin 2x = 2\sin x\cos x$$

$$\Rightarrow \sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

$$= \frac{R^2}{4}\int_{\Omega} d\Omega$$

$$= \frac{R^2}{4}(4\pi)$$

$$= \pi R^2 \text{ as expected.}$$

it is the cross sectional area of the sphere: BB's incident within this area will hit the target and those farther out will miss it completely.

- <u>scattering amplitude</u>

Consider an incident particle (spinless) of mass m that is being scattered by a static potential $V(\vec{r})$ of finite rang R, where the interaction between the incident particle and the potential occurs in a limited region ($r \leq R$). outside the rang ($r > R$), the pountential vanish ($V(\vec{r}) = 0$), and the eigenvalue problem reads $(\nabla^2 + k^2)\,\psi_{inc}(\vec{r}) = 0$; $k^2 = \frac{2mE}{\hbar^2}$

$\Rightarrow \psi_{inc}(\vec{r}) = A e^{i\vec{k}\cdot\vec{r}}$, where A is the normalization constant and $\vec{k}$ is the wave vector associated with the incident particle

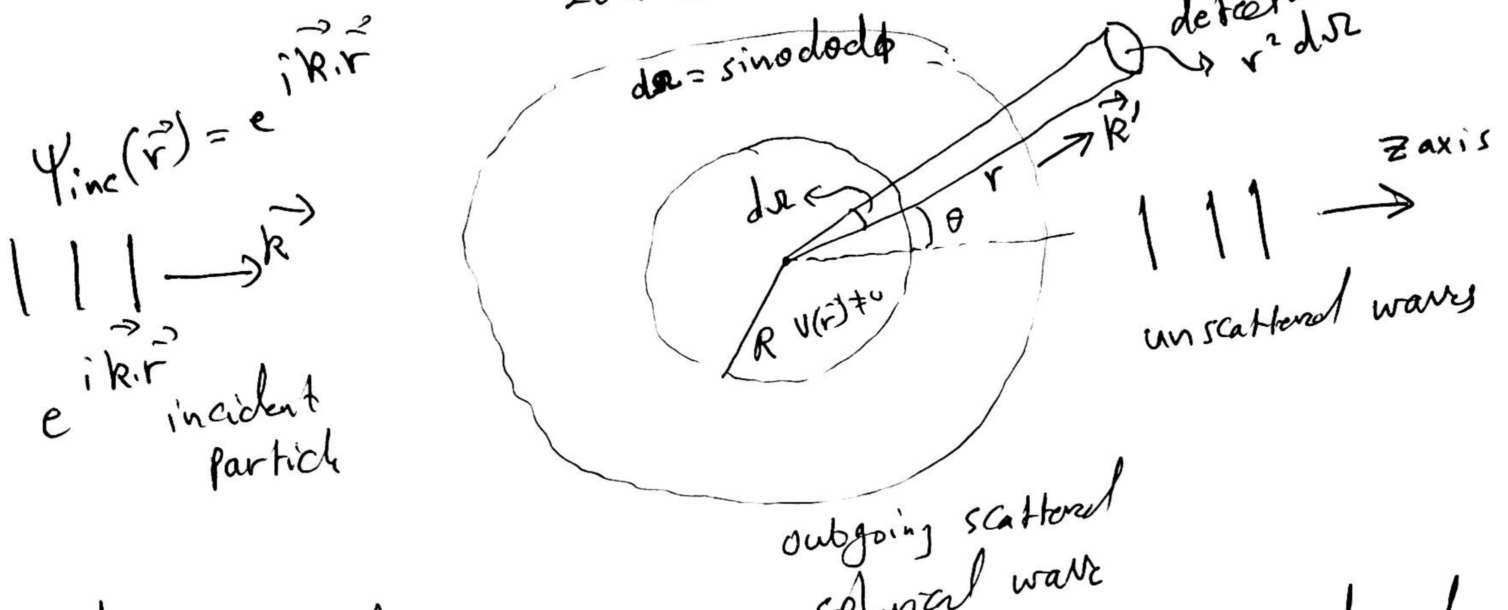
for simplicity, let us take $A = 1 \Rightarrow \psi_{inc}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

$$\vec{p} = \hbar\vec{k}$$

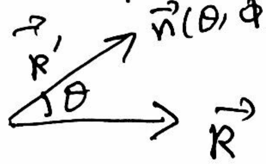
The incident flux $\vec{J}_{inc} = \frac{\hbar}{2im}\left\{\psi^*_{inc}\vec{\nabla}\psi_{inc} - \psi_{inc}\vec{\nabla}\psi^*_{inc}\right\}$

$$= \frac{\hbar}{2im}\left[e^{-i\vec{k}\cdot\vec{r}}\left(i\vec{k}e^{i\vec{k}\cdot\vec{r}}\right) - e^{-i\vec{k}\cdot\vec{r}}\left(-i\vec{k}\right)e^{i\vec{k}\cdot\vec{r}}\right\}$$

$$= \frac{\hbar}{2im}\left\{i\vec{k} + i\vec{k}\right\} = \frac{\hbar\vec{k}}{m}$$



now when the incident wave collides with the target, an outgoing spherical wave is produced. the wave vector of the scattered wave is $\vec{k}' = k\hat{r} = k\frac{\vec{r}}{r} = k\vec{n}$ where for elastic scattering $|\vec{k}'| = |\vec{k}|$



- the scattered spherical wave takes the form $\psi_{sc}(\vec{r}) = f(\vec{k},\vec{k}')\frac{e^{ikr}}{r}$; $f(\vec{k},\vec{k}') = f(\theta,\phi)$ is the scattering amplitude.

$$\vec{J}_{sc} = \frac{\hbar}{2im}\left\{\psi^*_{sc}\vec{\nabla}\psi_{sc} - \psi_{sc}\vec{\nabla}\psi^*_{sc}\right\}$$

$$= \frac{\hbar}{2im}\left[\left(f^*\frac{e^{-ikr}}{r}\right)f\left(\frac{1}{r}(ik)e^{ikr} - \frac{e^{ikr}}{r^2}\right) - c.c\right\}$$

$$= \frac{\hbar}{2im}\left\{|f|^2\left(\frac{1}{r^2}(ik) - \frac{1}{r^3}\right) - |f|^2\left(\frac{1}{r^2}(-ik) - \frac{1}{r^3}\right)\right\}$$

$$= \frac{\hbar}{2im}|f|^2\left\{\frac{ik}{r^2} - \cancel{\frac{1}{r^3}} + \frac{ik}{r^2} + \cancel{\frac{1}{r^3}}\right\} = \frac{\hbar}{2im}|f|^2\frac{2ik}{r} = \frac{\hbar k}{m}\frac{|f|^2}{r^2}$$

now recall that the # $dN(\theta,\phi)$ of particles scattered into an element of solid angle $d\Omega$ in the direction (θ,ϕ) is given by

$$dN = J_{sc} \underbrace{r^2 d\Omega}_{\text{area of detector}}$$

$$= \frac{\hbar k}{m} \frac{|f|^2}{\cancel{r^2}} \cancel{r^2} d\Omega = \frac{\hbar k}{m} |f|^2 d\Omega$$

now $$d\sigma = \frac{dN}{J_{inc}} = \frac{(\hbar k/m)\, |f|^2 d\Omega}{(\hbar k/m)} = |f|^2 d\Omega$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 \quad \text{differential cross section}$$

so the problem of determining the differential cross section is reduced to that of obtaining the scattering amplitude $f(\theta,\phi)$

total cross section $$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f|^2 d\Omega$$

now after the scattering has taken place, the total wave function at the detector consists of a superposition of ψ_{inc} and ψ_{sc}

$$\psi(\vec{r}) = \psi_{inc}(\vec{r}) + \psi_{sc}(\vec{r}) = \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{plane wave}} + \underbrace{f(\vec{k},\vec{k}')\frac{e^{ikr}}{r}}_{\text{spherical wave}} \quad (\text{for } r \gg R)$$

where $|f|^2$ is the probability of scattering in the direction (θ,ϕ)